Grade 11/12 Math Circles

November 30, 2022

Generating Functions 2 - Solutions

Exercise Solutions

Exercise: Rolling a regular 8-sided die

Create a combinatorial class (including a set and weight function) to represent rolling a regular 8-sided die.

Exercise: Rolling a regular 8-sided die Solution

Set: ways to roll the die $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Weight Function: $w(x) = x$, the value shown on the die.

Exercise: Choosing 5 cent coins

Create a combinatorial class (including a set and weight function) to represent choosing some number of 5 cent coins from an infinitely large pile.

Exercise: Choosing 5 cent coins Solution

Set: the ways that we could choose some number of coins $\{0, 1, 2, 3, 4, \ldots\}$.

Weight Function: $w(x) = 5x$, the total value of the coins chosen.

Exercise: Rolling two regular 4-sided dice - Method 1

Find the generating function for the combinatorial class representing one roll of two regular 4-sided dice, where the weight function is the sum of the values rolled. To find this generating function, create a chart of the potential results from rolling the two dice.

Exercise: Rolling two regular 4-sided dice - Method 1 Solution

Rolling two regular 4-sided dice gives the potential results:

	1	**2**	**3**	**4**
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

This gives a generating function of $z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8$.

Exercise: Rolling two regular 4-sided dice - Method 2

Find the generating function for the combinatorial class representing one roll of two regular 4-sided dice, where the weight function is the sum of the values rolled. This time, try using the multiplication of two combinatorial classes.

Exercise: Rolling two regular 4-sided dice - Method 2 Solution

Create a combinatorial class for rolling one regular 4-sided die:

$\mathcal{A} = \mathcal{B}$:
Set: Ways to roll a regular 4-sided die $\{1, 2, 3, 4\}$
Weight Function: $w(x) = x$
Generating Function: $z + z^2 + z^3 + z^4$

Then, if we multiply $\mathcal{A}$ and $\mathcal{B}$, we get:
Set: Ways to roll two 4-sided dice $\{(1, 1), (1, 2), \ldots, (4, 3), (4, 4)\}$
Weight Function: $w((a, b)) = a + b$
Generating Function:$(z + z^2 + z^3 + z^4)^2 = z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8$.

Exercise: Compositions of 4

What are the compositions of 4?

Exercise: Compositions of 4 Solution

The compositions of 4 are:

1 part: (4)

2 parts: (3, 1), (1, 3), (2, 2)

3 parts: (2, 1, 1), (1, 2, 1), (1, 1, 2)

4 parts: (1, 1, 1, 1)

Problem Set Solutions

1. Create combinatorial classes and corresponding generating functions for the following situations:

   (a) 0-1 strings where we wish to count by the length of the strings.

   (b) 0-1-2 strings where we wish to count by the length of the strings.

   (c) Strings with k possible number entries, where we wish to count by the length of the strings.

   (d) Drawing socks out of a basket, where there are 3 red socks, 5 blue socks, 10 purple socks and 12 green socks.

Solution:

(a) **Set:** 0-1 strings.

Weight Function: $w(x) =$ the length of the string.

Generating Function: $F(z) = 1 + 2z + 4z^2 + 8z^3 + \ldots = \sum_{n=0}^{\infty}(2z)^n = \frac{1}{1-2z}$

(b) **Set:** 0-1-2 strings.

Weight Function: $w(x) =$ the length of the string.

Generating Function: $F(z) = 1 + 3z + 9z^2 + 27z^3 + \ldots = \sum_{n=0}^{\infty}(3z)^n = \frac{1}{1-3z}$

(c) **Set:** Strings with k possible number entries.

Weight Function: $w(x) =$ the length of the string.

Generating Function: $F(z) = 1 + kz + k^2z^2 + k^3z^3 + \ldots = \sum_{n=0}^{\infty}(kz)^n = \frac{1}{1-kz}$

(d) Note that for this set, since we have not specified a weight function, our definition of the combinatorial class is not unique. The following includes an example of a valid weight function for the situation.

Set: The set of socks that could be drawn. For each sock that is the same colour, give it a different label so that it is different in the set.

Weight Function: $w(x) = 1$ if the sock is red, $w(x) = 4$ if the sock in blue, $w(x) = 10$ if the sock is purple and $w(x) = 15$ if the sock is green. (Note that we could have chosen any weights in the non-negative integers, $\mathbb{N}$, for the weight function since the required weights are not specified in the problem.)

Generating Function: $F(z) = 3z + 5z^4 + 10z^{10} + 12z^{15}$

2. Recall that $[z^n]F(z)$ represents the coefficient of z^n in the generating function $F(z)$. From your answers to Problem 1, find an expression for the following coefficients and describe each coefficient represents:
   (a) $[z^{12}]F(z)$, where $F(z)$ is the generating function from Problem 1a.
   (b) $[z^{40}]F(z)$, where $F(z)$ is the generating function from Problem 1b.
   (c) $[z^n]F(z)$, where $F(z)$ is the generating function from Problem 1c.

Solution: From the summation notation of the generating function:

(a) We see that the number in front of z^{12}, ie $[z^{12}]F(z)$, is 2^{12}.

(b) We see that $[z^{40}]F(z) = 3^{40}$.

(c) We get that the general form is $[z^n]F(z) = k^n$.

3. Find a generating function for 0-1-2 strings which start with 012.

Solution: Define $\mathcal{A}$ and $\mathcal{B}$ as the following:

$\mathcal{A}$:
Set: 0-1-2 strings
Weight Function: Length of string
GF: $A(z) = \frac{1}{1-3z}$

$\mathcal{B}$:
Set: $\{012\}$
Weight Function: Length of string
GF: $B(z) = z^3$

Then:

$\mathcal{B} * \mathcal{A}$:

Set: $(012, a)$, where a is a 0-1-2 string. This is equivalent to all 0-1-2 strings with 012 at the start of the string.

Weight Function: $w((012, a)) = 3 + w_A(a) =$ the length of the string created by pasting together all components.

Generating Function: $B(z)A(z) = \frac{z^3}{1-3z}$

4. Find an expression for the number of ways to make \$2.65 in change (with 5, 10 and 25 cent coins available, as well as 1 and 2 dollar coins).

Solution: We answer this in a similar manner to the coin problem. Define the following:

$\mathcal{A}$:
Set:
Choices for the number of 5¢ coins, $\{0, 1, 2, \ldots\}$
WF: $w_A(x) = 5x$
GF: $A(z) = \frac{1}{1-z^5}$

$\mathcal{B}$:
Set:
Choices for the number of 10¢ coins, $\{0, 1, 2, \ldots\}$
WF: $w_B(x) = 10x$
GF: $B(z) = \frac{1}{1-z^{10}}$

$\mathcal{C}$:
Set:
Choices for the number of 25¢ coins, $\{0, 1, 2, \ldots\}$
WF: $w_C(x) = 25x$
GF: $C(z) = \frac{1}{1-z^{25}}$

$\mathcal{D}$:
Set:
Choices for the number of \$1 coins, $\{0, 1, 2, \ldots\}$
WF: $w_D(x) = 100x$
GF: $D(z) = \frac{1}{1-z^{100}}$

$\mathcal{E}$:
Set:
Choices for the number of \$2 coins, $\{0, 1, 2, \ldots\}$
WF: $w_E(x) = 200x$
GF: $E(z) = \frac{1}{1-z^{200}}$

Then,
$\mathcal{A} * \mathcal{B} * \mathcal{C} * \mathcal{D} * \mathcal{E}$:
Set: (a, b, c, d, e). In other words, each element represents a choice of coins.
Weight Function: $w((a, b, c, d, e)) = 5a + 10b + 25c + 100d + 200e$. In other words, the total value of the coins chosen.
Generating Function: $A(z)B(z)C(z)D(z)E(z) = \frac{1}{(1-z^5)(1-z^{10})(1-z^{25})(1-z^{100})(1-z^{200})}$
To get our required result, we need to find the coefficient of z^{265}, as \$2.65 is 265 cents. So, an expression for the number of ways to make \$2.65 in change is $[z^{265}]\frac{1}{(1-z^5)(1-z^{10})(1-z^{25})(1-z^{100})(1-z^{200})}$. *Note: Using SageMath, we find there are 258 ways.*

5. Find two 4-sided dice such that:
   - Each side has a positive integer number of dots
   - The two dice are not the same
   - The probability of rolling a sum of $2, \ldots, 8$ on these dice is the same as the probabilities for regular 4-sided dice

Hint: $(z + z^2 + z^3 + z^4)^2 = (z^2 + 1)^2(z + 1)^2 z^2$

Solution: As with the Sicherman Dice, we start by converting each of these requirements into generating function terminology.

(a) Having a positive integer number of dots means that the new dice cannot have a constant (z^0) term.

(b) To have different dice, the corresponding generating functions for each die must be different.

(c) To have the same probabilities, the generating function created by multiplying the two dice generating functions together must be the same as for regular dice.

(d) To be 4-sided, $A(1) = B(1) = 4$.

From Problem 2b, we found that the generating function of rolling two 4-sided dice is $F(z) = z^2 + 2z^3 + 3z^4 + 4z^5 + 3z^6 + 2z^7 + z^8 = (z + z^2 + z^3 + z^4)^2$. If we factor this, we get $F(z) = (z^2 + 1)^2(z + 1)^2z^2$.

In order to satisfy Requirement (a), we need each die to have a factor of z. Since we need the dice to be 4-sided and $(1^2 + 1) = (1 + 1) = 2$, we need each die to have two total of the $z^2 + 1$ and $z + 1$ factors. Since the dice cannot be the same, this gives new dice with generating functions $A(z) = z(z+1)^2 = z+2z^2+z^3$ and $B(z) = z(z^2+1)^2 = z+2z^3+z^5$. So, our two new dice have sides $\{1, 2, 2, 3\}$ and $\{1, 3, 3, 5\}$.

6. **Challenge:** For any two n-sided fair dice ($n \geq 2$), will there always exist two other n-sided dice such that:

- Each side has a positive integer number of dots
- The two dice are not the same
- The probability of rolling a sum of $2, 3, 4, \ldots, 2n$ on these dice is the same as the probabilities for two fair n-sided dice

Solution: No. Consider $n = 2$. Then, in order to have the same probabilities of rolling $2, 3, 4$, the new dice must have generating functions which multiply to $(z + z^2)^2$ (similar to our solution for Problem 5). Factoring, this gives $(z + 1)^2z^2$. Since each side of our dice needs to have a positive integer number of dots, each die's corresponding generating function must have a factor of z. This leaves two remaining options for the generating functions of the dice.

Option 1: Die 1 and 2 have the same generating function, $(z + 1)z$. This is not valid as

it would imply that the two dice are the same.
Option 2: One die has a generating function of z and the other has a generating function $(z+1)^2 z$. This does not work as z would imply that the first die has only one side.
So, since no option works and there are no other dice for the $n = 2$ case. Thus, the statement does not hold in general.

7. How many compositions of n have k parts, where each part is an odd number?

Solution: We begin by defining the following combinatorial class:
$\mathcal{A}$:
Set: $\{1, 3, 5, \ldots\}$
Weight Function: $w(x) = x$
Generating Function: $A(z) = z + z^3 + z^5 + \ldots = \frac{z}{1-z^2}$
We note that $\mathcal{A}$ defines the combinatorial class for the compositions of n with 1 part, where the part is an odd number.
Then,
$\mathcal{A}^k$:
Set: $(a_1, a_2, \ldots, a_k)$, where each a_i is an element of $\{1, 3, 5, \ldots\}$
Weight Function: $w((a_1, a_2, \ldots, a_k)) = a_1 + a_2 + \ldots + a_k$
Generating Function: $(A(z))^k = \frac{z^k}{(1-z^2)^k}$

This then gives the generating function for the compositions with k parts, with each part an odd number, where the weight function is the sum of the integers in the composition.
So, we want:

$$\begin{aligned}
[z^n]\frac{z^k}{(1-z^2)^k} &= [z^n]z^k\frac{1}{(1-z^2)^k} \\
&= [z^{n-k}]\frac{1}{(1-z^2)^k} \\
&= [z^{n-k}]\sum_{m=0}^{\infty}\binom{m+k-1}{k-1}z^{2m} \\
&= \begin{cases} \binom{\frac{n-k}{2}+k-1}{k-1} & \text{if } n-k \text{ is even} \\ 0 & \text{if } n-k \text{ is odd} \end{cases}
\end{aligned}$$

8. Find the generating function for compositions of n which have k parts, where each part is at most 3.

Solution: We begin by defining the following combinatorial class:
$\mathcal{A}$:
Set: $\{1, 2, 3\}$
Weight Function: $w(x) = x$
Generating Function: $A(z) = z + z^2 + z^3$
We note that $\mathcal{A}$ defines the combinatorial class for the compositions of n with 1 part, where that part is at most 3.

Then,
$\mathcal{A}^k$:
Set: $(a_1, a_2, \ldots, a_k)$, where each a_i is an element of $\{1, 2, 3\}$
Weight Function: $w((a_1, a_2, \ldots, a_k)) = a_1 + a_2 + \ldots + a_k$
Generating Function: $(A(z))^k = (z + z^2 + z^3)^k$

This then gives the generating function for the compositions with k parts, with each part at most 3, where the weight function is the sum of the integers in the composition.

9. Find the generating function for compositions of n which have 1 or 2 parts.

Solution: We begin by defining the following combinatorial classes:
$\mathcal{A}$:
Set: $\{1, 2, 3, \ldots\}$
Weight Function: $w(x) = x$
Generating Function: $A(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z}$
We note that $\mathcal{A}$ defines the combinatorial class for the compositions of n with 1 part.

$\mathcal{A}^2$:
Set: (a_1, a_2), where a_1 and a_2 are each elements of $\{1, 2, 3, \ldots\}$
Weight Function: $w((a_1, a_2)) = a_1 + a_2$
Generating Function: $(A(z))^2 = \frac{z^2}{(1-z)^2}$
We note that $\mathcal{A}^2$ defines the combinatorial class for the compositions of n with 2 parts.

Then,
$\mathcal{A}^2 + \mathcal{A}$:
Set: (a_1, a_2), where a_1 and a_2 are each elements of $\{1, 2, 3, \ldots\}$ combined with $\{1, 2, 3, \ldots\}$
Weight Function:

$$\begin{cases} w((a_1, a_2)) = a_1 + a_2 & \text{if the element has two parts} \\ w(x) = x & \text{if the element has one part} \end{cases}$$

Generating Function: $A(z) + (A(z))^2 = \frac{z}{1-z} + \frac{z^2}{(1-z)^2}$

This then gives the generating function for the compositions with 1 OR 2 parts, where the weight function is the sum of the integers in the composition.

10. **Challenge:** How many compositions of n are there (of any number of parts)?

Solution: We begin by defining the following combinatorial class:
$\mathcal{A}$:
Set: $\{1, 2, 3, \ldots\}$
Weight Function: $w(x) = x$
Generating Function: $A(z) = z + z^2 + z^3 + \ldots = \frac{z}{1-z}$

We note that $\mathcal{A}$ defines the combinatorial class for the compositions of n with 1 part.

Then, similar to Problem 9, we wish to add together all possible classes which represent having a certain number of parts. In other words, we want $1 + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \mathcal{A}^4 + \ldots$, as each power of $\mathcal{A}$ represents adding another possibility for the number of parts in the composition (the 1 represents a composition with no parts).

Note that $1 + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \ldots = \sum_{k=0}^{\infty} \mathcal{A}^k$. We can thus apply a rule similar to the sum of Geometric Series to get that $\sum_{k=0}^{\infty} \mathcal{A}^k = \frac{1}{1-\mathcal{A}}$. So, our new combinatorial class has:
Set: Compositions of n
Weight Function: $w((a_1, a_2, \ldots)) = a_1 + a_2 + \ldots$
Generating Function: $\frac{1}{1-A(z)} = \frac{1}{(1-\frac{z}{1-z})}$

Then,

$$\begin{aligned}
[z^n]\frac{1}{\left(1-\frac{z}{1-z}\right)} &= [z^n]\frac{1}{\left(1-\frac{z}{1-z}\right)}\cdot\frac{1-z}{1-z} \\
&= [z^n]\frac{1-z}{1-2z} \\
&= [z^n]\left[\frac{1}{1-2z}-\frac{z}{1-2z}\right] \\
&= [z^n]\left[\sum_{n=0}^{\infty}(2z)^n - z\sum_{n=0}^{\infty}(2z)^n\right] \\
&= [z^n]\sum_{n=0}^{\infty}2^n z^n - [z^n]z\sum_{n=0}^{\infty}(2z)^n \\
&= [z^n]\sum_{n=0}^{\infty}2^n z^n - [z^{n-1}]\sum_{n=0}^{\infty}(2z)^n \\
&= 2^n - 2^{n-1} \\
&= 2^{n-1}(2-1) \\
&= 2^{n-1}
\end{aligned}$$